CHAPTER 6

LAPLACE TRANSFORM

INTRODUCTION

Many problems in engineering and physics can be described in terms of the evolution of solutions of linear differential equations subject to initial conditions. An important group of these problems involves constant coefficient differential equations, and equations like these can be solved very easily by using the Laplace transform.

The Laplace transform F(s) of a real function f(t) with domain of definition $0 \le t < \infty$ is defined as the integral transform

$$F(s) = \int_0^\infty e^{-st} \, f(t) \, dt \qquad (1)$$

where in general s , is a complex variable.

The importance of the Laplace transform in the study of initial value problems for linear constant coefficients differential equations is that it replaces the operation of integrating a differential equation in f(t) by much simpler algebraic operations involving F (s).

Unlike previous methods, where first a general solution is found, and then the constants in the complementary function are chosen

to match the initial conditions, when the Laplace transform method is used the initial conditions are incorporated from the start .
The task of finding the function f(t) from its Laplace transform F(s) is called inverting the transform, and when working with constant coefficients equations we can accomplish this by using the tables of Laplace transform pairs, that is, to a table listing a function f(t) and its corresponding Laplace transform F (s).

LAPLACE TRANSFORM: FUNDAMENTAL IDEAS►

The reason for interest in integral transforms in general, and the Laplace transform in particular, will become clear when the solution of initial value problems for differential equations is considered. It will then be seen that the Laplace transform replaces integrations with respect to t by simple algebraic operations involving F(s), so provided f(t) can be recovered from F(s) in a simple manner, the solution of an initial value problem can be found by means of straightforward algebraic operations.

Clearly the exponent e^{-st} will only decrease as t increases if $s > 0$, and the Laplace transform of f(t) will only be defined for functions f(t) that decrease sufficiently rapidly as $t \to \infty$ for the integral in (1) to exist . In general, if the function to be transformed is denoted by a lowercase letter such as f , then its Laplace transform will be denoted by the corresponding uppercase letter F, as in (1). It is convenient to denote the Laplace transform operation by the symbol *L*, so that symbolically

$$F(s) = L\{ f\{t) \} \qquad (2)$$

Definition 1 (The Laplace transform)

Let f (t) be defined for $0 \le t < \infty$. Then, the Laplace transform F(s) of f(t), written symbolically F(s) = L {f(t)}, is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt \qquad (3)$$

EXAMPLE 1►

Find

$$L\{1\} = \int_0^\infty e^{-st} dt$$

SOLUTION

$$L\{1\} = \int_0^\infty e^{-st} dt = [\frac{-1}{s} e^{-st}]_0^\infty = \frac{-1}{s}[e^{-st} - 1] = \frac{1}{s}$$

Find

$$L\{c\} = \int_0^\infty ce^{-st} dt$$

SOLUTION

$$L\{c\} = \int_0^\infty ce^{-st} dt = [\frac{-c}{s} e^{-st}]_0^\infty = \frac{-c}{s}[e^{-st} - 1] = \frac{c}{s}$$

Find $L\{e^{at}\}$ where a is real constant .

SOLUTION

Let $f(t) = e^{at}$ where a is constant. Then

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\, dt = \left. \frac{e^{-(s-a)t}}{a-s} \right|_0^{\infty}$$

$$L\left\{e^{at}\right\} = \frac{1}{s-a}, \qquad \text{provided } s > a$$

EXAMPLE 2► Find

$$L\left\{\frac{1}{t}\right\}$$

SOLUTION

$$L\left\{\frac{1}{t}\right\} = \int_0^{\infty} \frac{e^{-st}}{t}\, dt = \int_0^{1} \frac{e^{-st}}{t}\, dt + \int_1^{\infty} \frac{e^{-st}}{t}\, dt$$

But for t in the interval $0 \le t \le 1$, $e^{-st} \le 1$, if $s > 0$.

Thus, $$\int_0^{\infty} \frac{e^{-st}}{t}\, dt \ge e^{-st} \int_0^{1} \frac{dt}{t} + \int_1^{\infty} \frac{e^{-st}}{t}\, dt \quad .$$

However,

$$\int_0^{1} \frac{dt}{t} = \lim_{A \to 0} \int_A^{1} \frac{dt}{t} = \lim_{A \to 0} \ln[t]_A^1$$

$$= \lim_{A \to 0} (\ln 1 - \ln A) = \lim_{A \to 0} (-\ln A) = +\infty$$

Therefore, the integral (4) diverges, so that $f(t) = 1 / t$ has no Laplace transform . Example 2 and the preceding discussion pinpoint the need for answers to the following questions:

1. Which functions f have Laplace transforms?

2. Can two functions *f* and *g* have the same *Laplace transform*?

In order to give simple conditions that guarantee the existence of a Laplace transform, we shall require the following definition.

Definition 2► (Jump Discontinuity)

A function f has a jump discontinuity at a point t* if the function has different finite limits as it approaches t* from the left and from the right or if the two limits are equal but different from f(t*) Note that f (t*) may or may not be equal to either

$$\lim_{t \to t^{*+}} f(t) \quad or \quad \lim_{t \to t^{*-}} f(t)$$

In fact f(t*) may not even be defined.

Definition 3► (Piecewise Continuous Functions)

A function f defined on $(0,\infty)$ is piecewise continuous if it is continuous on every finite interval $0 \le t < \infty$, except possibly at finite many points in each such finite interval where it has jump discontinuities . Such a function is illustrated in Fig. 1. The class of piecewise continuous functions includes every continuous function as well as many important discontinuous functions, such as the unit step function, square waves, and the staircase function .

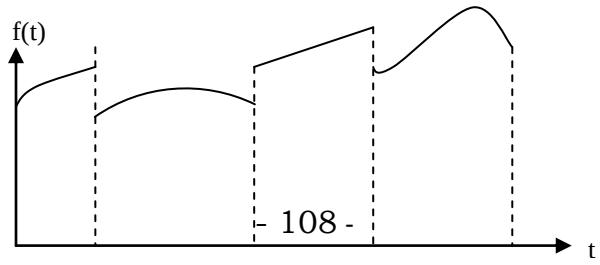


Fig 1

THEOREM 1► Existence Theorem

Let f be piecewise continuous on $t \geq 0$ and satisfy the condition

$$|f(t)| \leq Me^{at} \tag{6}$$

for t = T and for fixed nonnegative constants a, M , and T. Then

L{f{t)} = F(s) exists for all s > a.

PROOF

Since f is piecewise continuous, e^{-st} f(t) has a finite integral over any finite interval on $t \geq 0$, and

$$|L\{f(t)\}| = \left| \int_0^\infty e^{-st} f(t) dt \right| + \int_T^\infty e^{-st} |f(t)| \, dt \tag{7}$$

The first integral on the right–hand side of equation (6) exists, and by equation (5)

$$\int_0^\infty e^{-st} |f(t)| \, dt \leq M \int_T^\infty e^{-(s-a)t} \, dt = \frac{Me^{-(s-a)t}}{a-s} \Big|_T^\infty$$

which converges to M $e^{-(s-a)T}$ / (s – a) as $t \to \infty$, if s > a. Thus it follows by the comparison theorem of integrals that L { f(t) } exists. The conditions in **Theorem 1** are easy to test.

For example, $L\{t^n\}$ exists because

$$t^n < n!\ e^t, \quad \text{for any } t > 0,$$

To see this observe that

$$e^t = 1 + t + \frac{t^2}{2!} + ... + \frac{t^n}{n!} + ...$$

so that $t^n / n! = e^t$, for $t > 0$. On the other hand

$$t^2 > \ln M + a t$$

for sufficiently large t regardless of the choice of M and a. Take the exponent of both sides of this inequality, we get

$$e^{t^2} > e^{\ln M + at} = Me^{at}$$

It is not hard to show that e^{t^2} has no Laplace transform .

It should also be noted that there are functions having Laplace transform that do not satisfy the hypotheses of Theorem 1 . For example, $f(t) = 1 / \sqrt{t}$ is infinite at t = 0 ; but setting $x^2 = st$, we have

$$L\{t^{-1/2}\} = \int_0^\infty e^{-st} t^{-1/2} dt = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{s}}$$

One of the most important properties of the Laplace transform is stated in the following theorem.

THEOREM 2► (Linearity Property)

If L { a f(t)} and L {g(t)} exist, then L {a f(t) + b g(t)} exists and

$$L\{a f(t) + b g(t)\} = a L\{f(t)\} + b L\{g(t)\} \qquad (8)$$

PROOF

By definition

$$L\{af(t)+bg(t)\} = \int_0^\infty e^{-st}[af(t)+bg(t)]\,dt$$

$$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$$

$$= aL\{f(t)\} + b\{g(t)\} = aF(s) + bF(s)$$

The linearity property allows us to deal with a linear equation term by term in order to obtain its Laplace transform.

EXAMPLE 3►

Find Laplace transform of the following functions

i] f(t) = C

ii] f(t) = sin at

iii] f(t) = cos at .

SOLUTION

(i) f(t) = C

$$F(s) = \int_0^\infty e^{-st}\ C\ dt = C\frac{e^{-st}}{-s}\ \Big|_0^\infty = \frac{C}{s} \quad , \quad for\ s > 0$$

(ii) , (iii) Since e^{iat} = cos at + i sin at . Hence,

$$L\{e^{iat}\} = \int_0^\infty e^{-st}\ e^{iat}\ dt = \int_0^\infty e^{-(s-ia)t}dt\ = \frac{e^{-(s-ia)t}}{-(s-ia)}\Bigg|_0^\infty$$

$$= \frac{1}{s-ia} \times \frac{s+ia}{s-ia}$$

Thus

$$L\{e^{iat}\} = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

$$L\{\cos a\,t\} + i\,L\{\sin a\,t\} = = \frac{s}{s^2+a^2} + i\,\frac{a}{s^2+a^2}$$

Equating real and imaginary parts on both sides , we get

$L\{\cos a\,t\} = = \frac{s}{s^2 + a^2}$, and $L\{\sin a\,t\} = \frac{a}{s^2 + a^2}$

EXAMPLE 4 ►

Find Laplace transform of $f(t) = t^n$

SOLUTION

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt$$

To evaluate this integral we will make use of integration by parts to establish a recursion (recurrence) relation from which the result for arbitrary positive integer n can be found. Accordingly, we define

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

$u = t^n$ $\qquad$ $dv = e^{-st}dt$

$du = t^n$ $\qquad$ $v = \frac{-1}{s}e^{-st}$

$$L\{t^n\} = \left[\frac{-t^n}{s}e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$L\{t^n\} = \frac{n}{s}L\{t^{n-1}\}$$

$$L\{t^n\} = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.\ldots\ldots..L\{1\}$$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

In terms of Laplace transform notation , we have shown that

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad , \quad \text{for n = 0 , 1 , 2 , …} \qquad (9)$$

Notice that setting s = 1 in the general result of Example (4) enables n! to be expressed as the integral

$$n! = \int_0^\infty e^{-t}\ t^n\ dt \quad , \qquad for\ n = 0\,,1\,,2\,,....$$

This provides a way of representing factorial n in terms of an integral, and it is our first encounter with a special case of the Gamma function that will be required later.

the Gamma function . The gamma function, denoted by Γ(x) for x > 0, is defined by the integral

$$\Gamma(x) = \int_0^\infty e^{-t}\ t^{x-1}\ dt$$

In terms of the earlier notation, when the restriction that n is an integer is removed, and n is replaced by a positive real variable x, we can write

$$\Gamma(x+1) = \int_0^\infty e^{-t}\ t^x\ dt = I(x\,,1)$$

but

$$I(x\,,1) = x\ I(x-1\,,1) = x\ \Gamma(x) \quad , \quad for\ x > 0$$

so combining results shows that the gamma function satisfies the fundamental relation

$$\Gamma(x+1) = x\ \Gamma(x) \quad , \quad for\ x > 0\,. \qquad (10)$$

it is easily seen from this that

$$\Gamma(n+1) = n! \quad , \quad for\ n = 0, 1, 2, ... \tag{11}$$

so as $\Gamma(x))$ is defined for all positive x the gamma function provides a generalization of the factorial function n! for positive non-integer values of n. The gamma function, which belongs to the general class of functions called **higher transcendental functions**, occurs frequently throughout mathematics.

In Table 1 the transforms of some basic functions are listed .

TABLE 1: Laptace Transform Pairs

	F(s)	$L^{-1}\{F(s)\} = f(t)$
1	$\frac{1}{s}$	1
2	$1/s^2$	t
3	$\frac{1}{s^{n+1}}, n = 0 > 1, 2, ...$	$\frac{t^n}{n!}$
4	$\frac{1}{s-a}$	e^{at}
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$

DISCONTINUOUS FUNCTIONS►

Because the Laplace transform is defined in terms of an integral, it is possible to find Laplace transforms of discontinuous functions. Suppose, for example, that a function g(t) is discontinuous at t = a, as in Fig. 2 . Then, provided it converges, the integral defining the Laplace transform of g{t) is given by

$$L\{g(t)\} = \lim_{\varepsilon \to 0} \int_{0}^{a-\varepsilon} e^{-st} g(t)dt + \lim_{\delta \to 0} \int_{a+\delta}^{\infty} e^{-st} g(t)dt$$

where ε and δ are both positive. For simplicity, the upper limit in the first integral is usually denoted by $a^{(-)}$ and the lower limit in the second integral by $a^{(+)}$. These are, respectively, the limits of integration to the left and right of t = a.

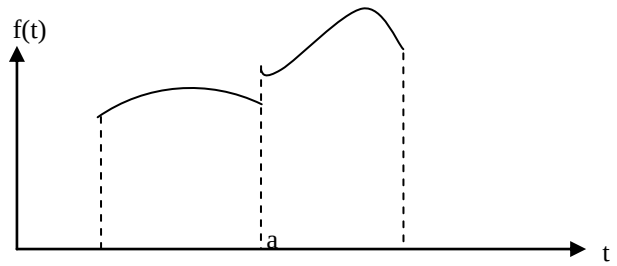


Fig. 2

OPERATIONAL PROPERTIES OF THE LAPLACE TRANSFORM►

In the previous section the Laplace transform of a basic list of commonly occurring functions f(t) was recorded as the list of Laplace transform pairs in Table 1. To use the Laplace transform to solve initial value problems for linear differential equations and

systems it is necessary to establish a number of fundamental properties of the transform known as its operational properties.

We now develop the most important operational properties of the Laplace transform, starting with the first shift theorem, also called the s–shift theorem.

THEOREM 3►[the first shift or the s-shift theorem **]**

Let L{f(t)} = F(s) for s > γ . Then the Laplace transform of $e^{a\,t}$ f(t) is obtained from F(s) by replacing s by s — a, where s — a > γ. Thus,

$$L\{e^{at} f(t)\} = F(s-a) \quad for \quad s-a > \gamma$$

Conversely , the inverse transform

$$L^{-1}\{F(s-a)\} = \quad e^{at} f(t)$$

PROOF

From the conditions of the theorem,

$$L\{e^{at} f(t)\} = \int_0^\infty e^{at} e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

Let $u = s - a$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-ut} f(t) dt = F(u) = F(s-a)$$

EXAMPLE 5►

Find L$\{e^{a\,t} t^n\}$, L$\{e^{a\,t}\ cos\ bt\}$, and L $\{e^{a\,t}\ t\ sin\ bt\}$.

SOLUTION

Using the Laplace transforms of t^n, cos bt, and t sin bt

$$L\{t^n\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad L\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}} \quad , \text{for } s > a$$

$$L\{\cos bt\} = \frac{s}{s^2+b^2} \quad \Rightarrow \quad L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2} \quad , s > a$$

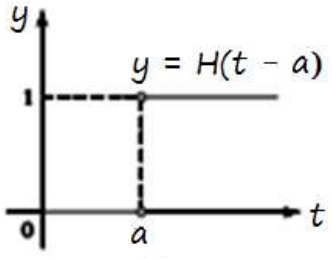
$$L\{\sin bt\} = \frac{b}{s^2+b^2} \quad \Rightarrow \quad L\{t\sin bt\} = -\frac{d}{ds}\left[\frac{b}{s^2+b^2}\right] = \frac{2bs}{[s^2+b^2]^2}$$

$$L\{e^{at}t\sin bt\} = \frac{2b(s-a)}{[(s-a)^2+b^2]} \quad , \quad s > a$$

HEAVISIDE STEP FUNCTION

An important discontinuous function that finds numerous applications in connection with the Laplace transform, and elsewhere, is the unit step function

$$f(t) = H(t-a) \;\; with \;\; a = 0$$



known also as the **Heaviside step function**. The unit step function is defined as

$$H(t-a) = \begin{cases} 0 & if\ t < a \\ 1 & if\ t > a \end{cases} \qquad (a \geq 0)$$

THEOREM 4► [The second shift or the t-shift theorem]

Let L[f(t)} = F(s). Then

$$L\{H(t-a)f(t-a)\} = e^{-sa}F(s)$$

EXAMPLE 6►
Prove that

$$L\{H(t-a)\} = \frac{e^{-as}}{s}, \quad for\ s > a \geq 0$$

PROOF
By definition

$$L\{H(t-a)\} = \int_0^\infty H(t-a)e^{-st}dt$$

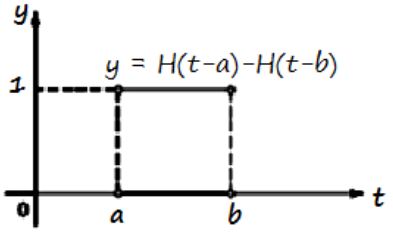
$$= \int_0^a H(t-a)e^{-st}dt + \int_a^\infty H(t-a)e^{-st}dt$$

$$= 0 + \int_a^\infty (1)e^{-st}dt \ = \left(-\frac{e^{-st}}{s}\right)_{t=a}^{\infty}$$

$$= \frac{e^{-as}}{s}, \quad for\ s > a \geq 0$$

UNIT PULSE FUNCTION
A related function that is also of considerable importance is the **unit pulse function**, defined as

$$p(t) = H(t-a) - H(t-b),$$
$$where \ \ b > a \geq 0$$



The function p(t) operates like a "switch," because it switches on at t = a and off at t = b. Graph of the function is shown in the Fig.

$$L\{H(t-a) - H(t-b)\} = \int_a^b e^{-st}dt \ = \int_a^\infty e^{-st}dt - \int_b^\infty e^{-st}dt$$

$$= \left(\frac{e^{-sa} - e^{sb}}{s}\right) \quad for\ s > b > a \geq 0$$

EXAMPLE 7►

Find L{g(t)} where

$$g(t)=\begin{cases}(t-2)^3 & ; \quad t>2\\ 0 & ; \quad t<2\end{cases}$$

SOLUTION

$$f(t)=(t-2)^3$$

$$L\{f\ (t-a)\}=e^{-as}F(s), \quad for\ s>a\geq 0$$

$$L\{t^3\}=\frac{3!}{s^4}=\frac{6}{s^4}=F(s)$$

$$\therefore L\{(t-2)^3\}=e^{-2s}.\frac{6}{s^4}=\frac{6e^{-2s}}{s^4}.$$

EXAMPLE 8►

Find $L\{H(t-4)\sin(t-4)\}$

SOLUTION

(a) We have $L\{\sin t\}=\dfrac{1}{s^2+1}$

So, applying the t – shift theorem gives

$$L\{H(t-4)\sin(t-4)\}=\frac{e^{-4s}}{s^2+1}$$

Laplace transform of the multiplication of f(t) by t ^{n}

THEOREM 5: Let $L\{f(t)\}=F(s)$, then

$$L\{t^n f(t)\}=(-1)^n\frac{d^nF(s)}{ds^n}$$

PROOF

By definition

$$\int_0^\infty e^{-st} f(t)\, dt = F(s)$$

so differentiating under the integral sign with respect to s gives

$$\frac{dF}{ds} = \int_0^\infty (-t) e^{-st} f(t)\, dt = -\int_0^\infty e^{-st}\ t\ f(t)\, dt$$

which is the result of the theorem when n = 1. Each subsequent differentiation will introduce a further factor (- t) into the integrand, leading the general result of the theorem.

EXAMPLE 9►

Find (a) $L\{t \sin at\}$ and (b) $L\{ t e^{at} \cos bt \}$.

SOLUTION

(a) Since $L\{ \sin at \} = a / \{s^2 + a^2\}$ for $s > 0$, hence

$$L\{t \sin at\} = (-1)\ \frac{d}{ds}\frac{a}{s^2 + a^2} = \frac{2as}{(s^2 + a^2)^2} \quad , \text{ for } s > 0,$$

(b) $L\{e^{at} \cos bt\} = (s - a) / [(s - a)^2 + b^2]$, $s > a$

Thus,

$$L\{t e^{at} \cos bt\} = -\frac{d}{ds}\frac{s - a}{[(s - a)^2 + b^2]}$$

$$= \frac{(s - a)^2 - b^2}{[(s - a)^2 + b^2]^2} \quad \text{for } s > a .$$

THEOREM 6 [Scaling theorem]

Let L (f(t)} = F(s). Then, if k > 0,

$$L\{f(kt)\} = \frac{1}{k} F(\frac{s}{k})$$

PROOF

The result follows by setting u = k t in the definition of the Laplace transform, because

$$L\{f(kt)\} = \int_0^\infty e^{-st} f(kt)dt = \frac{1}{k}\int_0^\infty e^{-s(u/k)} f(u)\, du$$

$$= \frac{1}{k} F(\frac{s}{k})$$

EXAMPLE 10►

If $L\{f(t)\} = e^{-3s}(1 - 2s) / (2s^2 - s + 1)$, find $L\{f(3t)\}$.

SOLUTION

In this case k= 3 > 0, so from Theorem 8 ,replacing s by s/3 in L{f(t)} and multiplying the result by 1/3 gives

$$L\{f(3t)\} = \frac{1}{3}\frac{e^{-s}(1-2s/3)}{[2(s/3)^2 - s/3 + 1]}$$

$$= \frac{e^{-s}(3-2s)}{2s^2 - 3s + 9}$$

THEOREM 11 [The transform of an integral]

Let f(t) be a piecewise continuous function such that

$$|f(t)| \le M\, e^{kt} \quad for\ k > 0\,.$$

If $L\{f(t)\} = F(s)$, then

$$L\left\{\int_0^t f(u)\, du\right\} = \frac{F(s)}{s} \quad , \quad for\ s > k$$

and, conversely,

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\, du$$

PROOF

The condition $|f(t)| \le M\, e^{kt}$ is sufficient to ensure the existence of the Laplace transform F(s). so writing $h(t) = \int_0^t f(u)\, du$ we have

$$|h(t)| \le \int_0^t |f(u)|\, du \le M \int_0^t e^{k\,u} du \le M\,\frac{e^{k\,t}}{k} \quad for \quad t \ge 0$$

This result shows that $|h(t)|$ grows no faster than $|f(t)|$ as $t \to \infty$, so the existence of the Laplace transform Y(s) ensures the existence of the Laplace transform of h(t) . Using the fundamental result from the calculus that h\`(t} = f(t) together with Theorem 2 means that. apart from points where f(t) is discontinuous,

$$F(s) = L\,\{f(t)\} = L\,\{h'(t)\} = s\,L\,\{h(t)\} = s\,L\left\{\int_0^t f(u)du\right\}$$

and so

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s} \quad , \quad for\ s > k$$

The converse result follows by taking the inverse Laplace transform.

EXAMPLE 11►

Find $L\left\{\int_0^t u\cos au\,du\right\}$

SOLUTION

(a) As L{ t cos at] = $(s^2 - a^2)/(s^2 + a^2)^2$ for s > 0,

$$L\left\{\int_0^t f(u)\,du\right\} = \frac{F(s)}{s} \quad , \quad for\ s > k$$

$$Put\ f(u) = u\cos au$$

$$L\left\{\int_0^t u\cos au\ \ du\right\} = \frac{s^2 - a^2}{s(s^2 + a^2)^2} \quad , \quad for\ s > 0$$

THEOREM 12 [Division by t]

Let f(t) / t be piecewise continuous, defined for $t \geq 0$ and such that $|f(t)/t| \leq M\,e^{kt}$ *for* $t \geq 0$. If $L\{f(t)/t\} = G(s)$ for s > k, and $L\{f(t)\} = F(s)$. Then

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

and. conversely,

$$L^{-1}\{G(s)\} = \frac{-1}{t}\,L^{-1}\{G'(s)\}$$

PROOF

We have

$$G(s) = \int_0^\infty e^{-s\,t}\,\frac{f(t)}{t}\,dt \quad for \quad s > k$$

However. from Theorem 7

$$G'(s) = \int_0^\infty e^{-s\,t}(-t)\frac{f(t)}{t}\,dt \quad = -\int_0^\infty e^{-st}\ f(t)\,dt = -F(s)$$

so after integration we have

$$\int_s^\infty F(u)\,du = -\int_s^\infty G'(u)du = G(s) - G(\infty)$$

To proceed further we now make use of the fact that the condition $|f(t)/t| \le M\,e^{kt}$ implies that $\lim_{s\to\infty} G(s) = 0$, showing that

$$G(s) = L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du \quad for\ s > k$$

The converse result follows by taking the inverse Laplace transform and using the fact that $L^{-1}\{G(s)\} = f(t)/t$ together with the result

$$L\ \{\ f(t)\ \} = F(s\} = -G'(s).$$

EXAMPLE 12►

Find $L\left\{\dfrac{\sin at}{t}\right\}$

SOLUTION

The function (sin at/t) is defined and finite for all t > 0. If we use the fact that

$$L\{\ \sin at\ \} = a\ /\ (s^2 + a^2),$$

it follows from that

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{u^2 + a^2}\, du$$

$$= \frac{\pi}{2} - Arc\tan\left(\frac{s}{a}\right)$$

$$= Arc\tan\left(\frac{a}{s}\right)$$

The theorem that follows shows how the initial values f(0) , f` (0) , ….. of a suitably differentiable function f(t) can be found directly from its Laplace transform F{s} .

EXAMPLE 13►

Compute the following integrals

$$(i) \quad \int_0^\infty t\, e^{-2t} \cos t\, dt \qquad (ii) \int_0^\infty \frac{e^{-t} - e^{-3t}}{t}\, dt$$

SOLUTION

$(i) \quad put \quad f(t) = t\cos t$

$$L\{f(t)\} = L\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) \qquad (a)$$

On the other hand , we have

$$L\{t\cos t\} = \int_0^\infty t\cos t\; e^{-st} dt \qquad (b)$$

From (a) and (b) we get

$$\int_0^\infty t\cos t\; e^{-st} dt = \frac{s^2 - 1}{(s^2+1)^2}$$

Put s = 2 , we obtain

$$\int_0^{\infty} t\, e^{-2t} \cos t \; dt = \frac{3}{25}$$

(*ii*) *Put* $f(t) = e^{-t} - e^{-3t}$

$$\therefore L\{f(t)\} = F(s) = \frac{1}{s+1} - \frac{1}{s+3} \qquad \text{(a)}$$

$$L\{\frac{f(t)}{t}\} = L\{\frac{e^{-t} - e^{-3t}}{t}\} = \int_s^{\infty} F(u)\, du$$

$$= \int_s^{\infty} (\frac{1}{u+1} - \frac{1}{u+3}) du$$

$$\therefore L\{\frac{e^{-t} - e^{-3t}}{t}\} = \ln \frac{s+3}{s+1}$$

Since, $$L\{\frac{e^{-t} - e^{-3t}}{t}\} = \int_0^{\infty} e^{-st} (\frac{e^{-t} - e^{-3t}}{t}) dt \qquad \text{(b)}$$

Equating (i) and (ii) we obtain

$$\therefore \int_0^{\infty} e^{-s\,t} (\frac{e^{-t} - e^{-3t}}{t}) dt = \ln \frac{s+3}{s+1}$$

To obtain the required result we take the limit as $s \to o^+$. Thus

$$\therefore \int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} dt = \ln(3) \ .$$

THEOREM 2 (Transforming derivatives)

Let f(t) be continuous on $0 \le t \le \infty$, and let $f'(t)$ be piece-wise continuous on every finite interval contained in $t \ge 0$. Then

$$L\{f'(t)\} = -f(0) + s\,F(s)$$

where

$$\mathrm{L}\{\ \mathrm{f(t)}\} \ = \mathrm{F(s)}$$

PROOF

Using integration by parts, and assuming that f satisfies the sufficiency condition for the existence of a Laplace transform, we have

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

$$L\{f`(t)\} = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-s)e^{-st} f(t)\,dt$$

$$L\{f`(t)\} = -f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$L\{f'(t)\} = -f(0) + s\,F(s)$$

where $e^{-st} f(t) \to 0 \quad as.\ t \to \infty$

THEOREM 7 (Transform of a higher derivative)

Let f(t) be continuous on $0 \le t \le \infty$, and let f` (t), f``(t),..., $f^{(n-1)}$ be piecewise continuous on every finite interval contained in $t \ge 0$. If L{ f(t) } = F(s} . Then

$$L\{f^{(n)}(t)\} = -f^{(n-1)}(0) - s\,f^{(n-2)}(0). - ..... - s^{n-1} f(0) + s^n F(s)$$

PERIODIC FUNCTIONS►

Many functions whose Laplace transform is required are periodic functions with period T , though they are not necessarily continuous functions for all t > 0 . In the Laplace transform, where only the behavior of a function f(t) for t > 0 is involved, a periodic func-

tion with period T is defined as a function f(t) with the property that T is the smallest value for which

$$f(t + T) = f(t) \text{ for all } t > 0.$$

THEOREM 8► [Transform of a periodic function with period T]

Let L{f(t)} = F(s) be a periodic function with period T such that

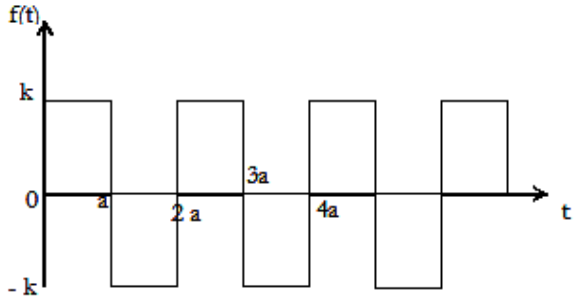
$$\int_0^T e^{-st} f(t)\, dt \quad \text{is finite.}$$

Then

$$L\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt \quad , \quad for \quad s > 0$$

EXAMPLE 13►

Find the Laplace transform of the square wave shown in Figure



SOLUTION

As the function is discontinuous with period 2a , we compute the integral in two parts as

$$\int_0^{2a} e^{-st} f(t)\,dt = \int_0^{a} k\,e^{-st}\,dt + \int_a^{2a} (-k)\,e^{-st}\,dt$$

$$= \frac{k}{s}(1-e^{-as}) + \frac{k}{s}(e^{-2as} - e^{-as})$$

$$= \frac{k}{s}(1 + e^{-2as} - 2\,e^{-as})$$

Then from Theorem (8) we have

$$L\{f(t)\} = \frac{k(1-2\,e^{-as}+e^{-2as})}{s(1-e^{-2as})}$$

$$= \frac{k(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})} = \frac{k\,(1-e^{-as})}{s\,(1+e^{-as})}$$

$$= \frac{k\,(e^{as/2} - e^{-as/2})}{s\,(e^{as/2}+e^{-as/2})} = \frac{k\sinh(as/2)}{s\cosh(as/2)}$$

$$= \frac{k}{s}\tanh(k\,s/2) \quad , \quad for\ s>0$$

THEOREM 9► **[**The initial value theorem**]** (without proof)

Let L{F(t)} =F(s) be the Laplace transform of an n -times differentiable function f(t). Then

$$f^{(r)}(0) = \lim_{s\to\infty}\{s^{r+1}F(s) - s^r f(0) - s^{r-1}f'(0) - \ldots - sf^{(r-1)}(0)\}$$

r =0 , 1 , 2 ,….

In particular,

$$f(0) = \lim_{s\to\infty}\{sF(s)\} \quad , \quad f'(0) = \lim_{s\to\infty}\{s^2F(s) - s\,f(0)\}$$

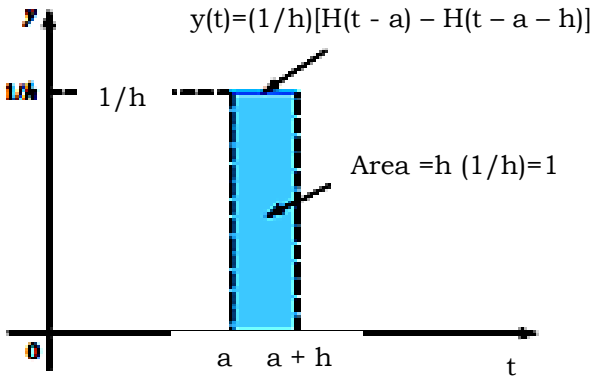
$$f''(0) = \lim_{s\to\infty}\{s^3F(s) - s^2\,f(0) - s\,f'(0)\}$$

The last operational property to be considered concerns the **Dirac delta function**, usually abbreviated to the **delta function** and

sometimes called the **unit impulse function**. The Dirac delta function denoted by

δ (t — a), is actually a limiting mathematical operation, and not a function as its name implies. For our purposes the delta function can be considered to be the limit of a rectangular "'pulse" of height h and width 1/h in the limit as $h \to \infty$. Thus the area of the graph representing the pulse remains constant at 1 as $h \to \infty$, while its height increases to infinity and its width decreases to zero.

The graphical representation of such a pulse f(t) = δ(t – a) located at t = a , before proceeding to the limit , is shown in Figure below



Definition The delta function►

The delta function located at t = a and denoted by

δ (t — a) is defined as the limit

$$\delta(t-a) = \lim_{h \to 0} \frac{1}{h}[H(t-a) - H(t-a-h)]$$

The operational property of the delta function, usually called its **filtering property** and sometimes its shifting property. is represented by the following theorem.

THEOREM 10► [Filtering property of the delta function]

Let f(t) be defined and integrable over all intervals contained within $0 \le t < \infty$, and let it be continuous in a neighborhood of a. Then for $a \ge 0$

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$$

EXERCISES	
Find Laplace transform of the following functions	
1. f(t) = 2 sin 3t –cos3 t.	2. f(t) = t –t^2 + t^3.
3. f(t) = e^{3t} (sin t – cos t)	4. f(t) = e^{-2t}(cos 2t – sin 2t).
5. f(t) = t (sin 2t – cos2t)	6. f(t) = t cosh 3t – sinh3 t.
7. f(t) = sinh t – t cos t	8. f(t) = e^{-t} cos2t –t
9. f(t) = 2 t^2 – 3 t + 4 cos 3t	
10. Find $L\{t^2e^{3t}\sin t\}$. 11. Find $L\{te^{-t}\sin 4t\}$.	

INVERSE OF LAPLACE TRANSFORM►

The two functions f(t) and F(s) are called a Laplace transform pair, and for all ordinary functions, given F(s) the corresponding function f(t) is determined uniquely, just as f(t) determines F(s) uniquely. This relationship is expressed symbolically by using the symbol L^{-1} to denote the operation of finding a function f(t) with a given Laplace transform F(s). This process is called finding the **inverse Laplace transform of F(s)**. In terms of the foregoing example, we have

$$L\{e^{at}\} = 1/(s-a) \implies L^{-1}\{1/(s-a)\} = e^{at} \qquad (1)$$

This is a particular case of the general result that , by definition, the inverse Laplace transform acting on the Laplace transform of the function returns the original function, so we can write

$$L^{-1}\{L\{ f(t) \}\} = f(t)$$

COROLLARY►

(Linearity Property of the Inverse Laplace Transform)

If $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$,

Then

$$L^{-1}\{a F(s) + b G(s)\} = a L^{-1}\{F(s)\} + b L^{-1}\{G(s)\}$$
$$= a f(t) + b g(t).$$

The linearity property allows us to deal with a linear equation term by term in order to obtain its Laplace transform.

The process of finding an inverse Laplace transformation involves reversing the foregoing argument and seeking a function f(t} that

has the required Laplace transform F(s). Where possible, this is accomplished by simplifying the algebraic structure of F(s) to the point at which it can be recognized as the sum of the Laplace transforms of known functions of t.

Rules of Partial Fractions

1. The numerator must be of lower degree than the denominator . If it is not , then we first divide out .
2. Factorize the denominator into its prime factors . These determine the shape of the partial fraction .
3. A linear factor (s + a) gives a partial fraction $\frac{A}{s+a}$, where A is a constant to be determined
4. A repeated factor $(s + a)^2$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2}$
5. Similarly $(s + a)^3$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$
6. A quadratic factor $(s^2 + p s + q)$ gives $\frac{P s + Q}{s^2 + p s + q}$
7. Repeated quadratic factors $(s^2 + p s + q)^2$ gives

$$\frac{P s + Q}{s^2 + p s + q} + \frac{R s + T}{(s^2 + p s + q)^2}$$

So , $\frac{s - 19}{(s+2)(s-5)}$ has partial fractions of the form

$$\frac{A}{(s+2)} + \frac{B}{(s-5)}$$

and $\dfrac{3s^2 - 4s + 11}{(s+3)(s-2)^2}$ has partial fraction of the form

$$\frac{A}{(s+3)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

EXAMPLE 1►

Find the inverse Laplace transform of

$$F(s) = \frac{5s+1}{s^2 - s - 12}$$

Solution

Expanding the Laplace transform in terms of partial fractions

$$\frac{5s+1}{s^2 - s - 12} = \frac{5s+1}{(s-4)(s+3)}$$

Then the partial fractions are of the form

$$\frac{A}{(s-4)} + \frac{B}{(s+3)}$$

If we multiply through by the denominator ($s^2 - s - 12$) we have

$$5s + 1 = A(s + 3) + B(s - 4)$$

(i) Let $s = 4$, then $21 = 7A \implies A = 3$

(ii) Let $s = -3$, then $-14 = -7B \implies B = 2$

$$\frac{5s+1}{s^2 - s - 12} = \frac{3}{(s-4)} + \frac{2}{(s+3)}$$

Hence,

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\{\frac{3}{(s-4)}\} + L^{-1}\{\frac{2}{(s+3)}\}$$

$$= 3\,e^{4t} + 2\,e^{-3t}$$

EXAMPLE 2►

Find the inverse transform of $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2}$.

SOLUTION

Expanding the Laplace transform in terms of partial fractions as:

$$\frac{A}{s+2} + \frac{B}{(s-3)} + \frac{C}{(s-3)^2}$$

Multiply through by $(s + 2)(s - 3)^2$ we get

$$s^2 - 15s + 41 = A(s-3)^2 + B(s+2)(s-3) + C(s+2)$$

Put $s = 3$, we get , $5C = 5$, then $C = 1$

Put $s = -2$, we get , $25A = 75$, then $A = 3$

Equating the coefficient of s^2 on both sides, we get

$B = -2$

So , $F(s) = \dfrac{s^2 - 15s + 41}{(s+2)(s-3)^2} = \dfrac{3}{s+2} + \dfrac{-2}{(s-3)} + \dfrac{1}{(s-3)^2}$

Now ,

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\{\frac{3}{s+2}\} + L^{-1}\{\frac{-2}{(s-3)}\} + L^{-1}\{\frac{1}{(s-3)^2}\}$$
$$= 3\,e^{-2t} - 2\,e^{3t} + t\,e^{3t}$$

EXAMPLE 3 ► Find

$$L^{-1}\{(s^4 + 5s^2 + 2) / [s^3(s^2 + 1)]\}.$$

SOLUTION

Simplifying the transform by means of partial fractions gives

To make further progress with the Laplace transform it is necessary to have available a table of Laplace transform pairs for the most commonly occurring functions . Theorems to be developed later will enable such a table to be extended in a straightforward manner, so that transforms and inverse Laplace transforms of more complicated functions can be found.

TABLE 1: Laptace Transform Pairs

	$F(s)$	$L^{-1}\{F(s)\} = f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^2}$	t
3	$\frac{1}{s^{n+1}}, n = 0 > 1, 2, \ldots$	$\frac{t^n}{n!}$
4	$\frac{1}{s-a}$	e^{at}
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
6	$\frac{s}{s^2+a^2}$	$\cos at$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
8	$\frac{s}{s^2-a^2}$	$\cosh at$

THEOREM 5 [the first shift or the s-shift theorem **]**

Let L{f(t)} = F(s) for s > γ . Then the Laplace transform of $e^{a\,t}$ f(t) is obtained from F(s) by replacing s by s — a, where s — a > γ. Thus

$$L^{-1}\{F(s-a)\} = \quad e^{at} f(t)$$

EXAMPLE 4 Find

$$L^{-1}\left\{\frac{1}{s^2+4s+13}\right\}$$

SOLUTION

Completing the square in the denominator we have

$$L^{-1}\left\{\frac{1}{s^2+4s+13}\right\} = L^{-1}\left\{\frac{1}{(s+2)^2+9}\right\} = \frac{1}{3}e^{-2t}\sin 3t$$

THEOREM 6► [The transform of an integral]

Let f(t) be a piecewise continuous function such that

$$|f(t)| \le M\, e^{kt} \quad \text{for } k > 0\,.$$

If $L\{f(t)\} = F(s)$, then

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\, du$$

EXAMPLE 5►

Find $\quad L^{-1}\left\{\dfrac{1}{s(s^2+a^2)}\right\}$

SOLUTION

We can write

$$\frac{1}{s(s^2+a^2)} = \frac{1}{s}\,\frac{1}{s^2+a^2}$$

So if we set

$$F(s) = \frac{1}{(s^2+a^2)}$$

for which

$$f(t) = L^{-1}\{F(s)\} = \frac{1}{a}\sin at$$

it follows that

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \int_0^t \frac{1}{a}\ sin\ a\ u\ du$$
$$= \frac{1-cos\ at}{a^2}$$

THEOREM 7► [Division by t]

Let f(t) / t be piecewise continuous, defined for $t \geq 0$ and such that $|f(t)/t| \leq M\, e^{kt}$ *for* $t \geq 0$. If $L\{f(t)/t\} = G(s)$ for s > k, and $L\{f(t)\} = F(s)$. Then

$$L^{-1}\{G(s)\} = \frac{-1}{t} L^{-1}\{G'(s)\}$$

EXAMPLE 6

Find $L^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\}$

SOLUTION

If we set

$$G(s) = \ln\left(\frac{s+a}{s+b}\right)$$

Differentiating gives

$$G'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

from which we see that

$$L^{-1}\{G'(s)\} = e^{-at} - e^{-bt}$$

Thus

$$L^{-1}\{G(s)\} = L^{-1}\left\{\ln\left(\frac{s+a}{s+b}\right)\right\} = -\frac{1}{t}L^{-1}\{G'(s)\}$$
$$= (e^{-bt} - e^{-at}) / t$$

EXERCISES

Find L^{-1} {F(s)} for the following functions

1. $F(s) = (s^2 - 1) / [s(s^2 + 4)]$.

2. $F(s) = (s^2 + 3s + 1) / [s(s^2 - 4)]$.

3. $F(s) = (3s + 5) / [s(s^2 + 9)]$.

4. $F(s) = (s^2 - 4) / [(s^2 + 1)(s^2 - 1)]$

5. $F(s) = (s^3 - 1) / [(s + 2)^2 (s^2 - 9)]$

6. $F(s) = (s^2 + s + 1) / [(s^2 + 4)(s^2 - 9)]$.

7. $F(s) = s^2 / [(s - 1)^2 (s + 1)]$.

8. $F(s) = s / (s - 1)^3$

9. $F(s) = (s^2 + 4) / [(s^2 - 9)(s - 1)]$.

10. $F(s) = (s^2 + 1)/[(s + 1)(s + 2)(s + 3)$.

11. Find $L^{-1}\{s\, e^{-2s}/(s^2 + 4)\}$.

12. Find $L^{-1}\{e-\pi s/3/(s^2 + 9)\}$.

13. Find $L^{-1}\{e-\pi s/2(s + 1)/(s^2 + 4s + 5)\}$.

14. Find $L^{-1}\{e^{-2s}(s^2 + s + 1)/[s(s + 2)2]\}$.

15. Find $L^{-1}\{e-4s(s + 3)/(s^2 + 4s + 13)\}$.

16. Find $L^{-1}\{e^{-3s}\, s^2/[s(s^2 + 4s + 8)]\}$.

17. If $L\{ F(s)\} = 1/[s^2(s^2 + 4)]$, find $L\{ f(t/3)\}$.

18. If $L\{ F(s)\} = (s^2 - 4)/[(s^2 + 4)^2]$, find $L\{ f(t/2)\}$.

APPLICATIONS ON LAPLACE TRANSFORM IN SOLVING OF INITIAL VALUE PROBLEMS

To solve a differential equation by Laplace transform , we go through four distinct stages .

(i) Re–write the equation in terms of Laplace transform

(ii) Insert the given initial conditions

(iii) Rearrange the equation algebraically to give the transform of the solution

(iv) Determine the inverse transform to obtain the particular solution

EXAMPLE 1: Solve the initial value problem

$$y'' + y = t \qquad , y(0) = 1 \qquad , y'(0) = -2$$

SOLUTION

Take Laplace transform of both sides

$$L\{y''\} + L\{y\} = L\{t\}.$$

$$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0) = sY(s) - s + 2$$

$$L\{y\} = Y(s)$$

$$L\{t\} = \frac{1}{s^2}$$

$$2 - s + s^2Y + Y = \frac{1}{s^2}$$

$$(s^2 + 1)Y = \frac{1}{s^2} + s - 2$$

$$Y(s) = \frac{1}{s^2(s^2 + 1)} + \frac{s}{s^2 + 1} - \frac{2}{s^2 + 1}$$

$$Y(s) = \frac{1}{s^2} + \frac{s}{s^2 + 1} - \frac{3}{s^2 + 1}$$

$$\therefore y(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + L^{-1}\left\{\frac{s}{s^2 + 1}\right\} - 3L^{-1}\left\{\frac{1}{s^2 + 1}\right\}$$

$$y(t) = t + \cos t - 3\sin t.$$

EXAMPLE 2 Solve the differential equation

$$y'' + 3y' + 2y = 4e^{2t} \qquad , y(0) = -3 \qquad , y'(0) = 5$$

SOLUTION

$$L\{y''\} - 3L\{y'\} + 2L\{y\} = 4L\{e^{2t}\}$$

$$(s^2Y - sy(0) - y'(0)) - 3[sY - y(0)] + 2Y = \frac{4}{s - 2}$$

$$(s^2Y + 3s - 5) - 3(sY + 3) + 2Y = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)Y = \frac{4}{s-2} + 14 - 3s$$

$$\therefore Y(s) = \frac{4}{(s-2)(s-2)(s-1)} + \frac{14-3s}{(s-2)(s-1)}$$

$$= \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

Using partial fractions, we get

$$Y(s) = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

Take the inverse of the transform, we get

$$\therefore y(t) = L^{-1}\{Y(s)\} = -7L^{-1}\left\{\frac{1}{s-1}\right\} + 4L^{-1}\left\{\frac{1}{s-2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

$$= -7e^t + 4e^{2t} + 4L^{-1}\left\{\frac{d}{ds}\left(\frac{-1}{s-2}\right)\right\}$$

$$= -7e^t + 4e^{2t} + 4te^{2t}$$

EXAMPLE 3 Solve the initial value problem

y'' + 3 y' + 2y = sin2t , where y(0) =2 and y`(0) = –1.

SOLUTION

Taking the Laplace transform of the differential equation we have

L { y'' } + 3 L{ y ' } + 2 L{ y } = L[sin2t}.

Setting L {y(t)} = Y(s), and using the initial conditions

y(0) = 2 and y` (0) = –1,

We find that

$$L\{y''\} = s^2Y(s) - sy'(0) - y(0)$$
$$= s^2Y(s) - 2s + 1$$

and

$$L\{y'\} = sY(s) - y(0)$$
$$= sY(s) - 2$$

$$L\{\sin 2t\} = \frac{2}{s^2 + 4}$$

so combining these results enables the transformed differential equation to be written

$$s^2Y(s) - 2s + 1 + 3[sY(s) - 2] + 2Y(s) = \frac{2}{s^2 + 4}$$
$$(s^2 + 3s + 2)Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{s^2 + 4}$$

Solving for the Laplace transform of the solution gives

$$Y(s) = \frac{2s^3 + 5s^2 + 8s + 22}{(s^2 + 4)(s^2 + 3s + 2)}$$

When expressed in partial fraction form, Y(s) becomes

$$Y(s) = \frac{-5}{4}\ \frac{1}{s+2} + \frac{17}{5}\ \frac{1}{s+1} - \frac{1}{20}\ \frac{2}{s^2+4} - \frac{3}{20}\ \frac{s}{s^2+4}$$

Using the linearity property when taking the inverse Laplace transform, we have

$$L^{-1}\{Y(s)\} = \frac{-5}{4}\ L^{-1}\{\frac{1}{s+2}\} + \frac{17}{5}\ L^{-1}\{\frac{1}{s+1}\}$$
$$- \frac{1}{20} L^{-1}\{\ \frac{2}{s^2+4}\} - \frac{3}{20} L^{-1}\{\ \frac{s}{s^2+4}\}$$

Using Table 1 to identify the four transforms involved shows that the solution of the initial value problem is

$$Y(s) = \frac{-5}{4}\ \frac{1}{s+2} + \frac{17}{5}\ \frac{1}{s+1} - \frac{1}{20}\ \frac{2}{s^2+4} - \frac{3}{20}\ \frac{s}{s^2+4}$$

$$y(t) = \frac{-5}{4}e^{-2t} + \frac{17}{5}e^{-t} - \frac{1}{20}\sin 2t - \frac{3}{20}\cos 2t \quad , \quad t > 0$$

EXAMPLE 4► Solve the initial value problem

$$y'' + 2y' + 5y = e^{-t}\sin t \qquad ; y(0) = 0\ , y'(0) = 1$$

SOLUTION

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}$$

$$[s^2Y - sy(0) - y'(0)] + 2[sY - y(0)] + 5[Y] = \frac{1}{(s+1)^2+1}$$

$$(s^2Y - 1) + 2sY + 5Y = \frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)Y(s) = \frac{1}{s^2+2s+2} + 1$$

$$Y(s) = \frac{1}{s^2+2s+5} + \frac{1}{(s^2+2s+2)(s^2+2s+5)}$$

$$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

Using partial fractions, we get

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{s^2+2s+2} + \frac{Cs+D}{s^2+2s+5}$$

$$= \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5}$$

$$= \frac{1/3}{(s+1)^2+1} + \frac{2/3}{(s+1)^2+4}$$

$$y(t) = L^{-1}\{Y(s)\} = \frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{2}{(s+1)^2+4}\right\}$$

$$y(t) = \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t = \frac{1}{3}e^{-t}(\sin t + \sin 2t)$$

EXERCISES

Solve the initial value problems

1. Solve $y'' \ 3y' + 2y = \cos t$, with $y(0)=1$, $y'(0)=-1$.

2. Solve $y'' + 5y' + 4y = \exp(-t)$, with $y(0)=1$, $y'(0)=0$.

3. Solve $y'' + 8y' - 9y = t$, with $y(0) = 2$, $y'(0) = 1$.

4. Solve $y'' + 5y' + 6y = 1 + t^2$, with $y(0)=0$, $y'(0)=0$.

5. Solve $y'' + 5y' + 6y = \cos(t)$, with $y(0)= 1$, $y'(0) = 0$.

6. Solve $y'' - 5y' + 6y = t$, with $y(0)=0$, $y'(0)= 0$.

7. Solve $y'' - 5y' + 6y = 1 + t$, with $y(0) = 0$, $y'(0) = 1$.

8. Solve $y'' - 6y' + 10y = t$, with $y(0) = 1, y'(0) = 1$.

9. Solve $y'' + 2y' + 10y = e^{-t}H(t-1)$, with $y(0) = -1, y'(0) = 0$.

10. Solve $y'' - y' - 2y = e^{-t}$ with $y(0) = 1, y'(0) = 0$.